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SOME IDENTITIES ON THE FULLY DEGENERATE BELL POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. Recently many interesting works are done on the degenerate Bell polynomials, which are intiated by T. Kim and D. S. Kim. And later T. Kim et al. studied on the partially degenerate Bell polynomials. Also D. V. Dolgy et al. defined and studied fully degenerate Bell polynomials. Recently, S. Pyo and T. Kim studied some identities on the fully degenerate Belly polynomials by using differential equations.

In the paper, we consider new denenerating approach to the Bell polynomials which is called fully degenerate Bell polynomials of the second kind. We study some identities from the generating function and specially by using the differential equations on those polynomials. We derive some identities of our fully degenerate Bell polynomials of the second kind, which related with higher order Daehee polynomials and higher order Bernoulli numbers.

1. INTRODUCTION

In combinatorial mathematics, the Bell polynomials denoted by $\mathrm{B}_n(x)$, are used in the study of partitions (see [1, 2, 4, 5, 8]). The Bell polynomials are defined by the generating function to be [7, 8, 11, 14]

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{B}_n(x) \frac{t^n}{n!}.$$

When $x = 1$, $\mathrm{B}_n = \mathrm{B}_n(1)$ are called the n-th Bell numbers.

As is well known, the Bell numbers are equal to the number of partitions of n-set. For $n \geq 0$, the Stirling numbers of the second kind $S_2(n, k)$ are defined as

$$x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k, \tag{1}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$.

From (1), we note that the generating function for $S_2(n, k)$ is given by

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n, k)\frac{t^n}{n!}, (k \geq 0).$$

For $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by (see [3, 6, 9, 10])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!},$$

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where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $(n \geq 1)$. In view of (1), T. Kim et al. defined the degenerate λ-Striling numbers of the second kind by the generating function, for $k \geq 0$

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \quad \text{(see [1, 12, 13])}.$$

Also the following relation is known in [6]

$$\begin{aligned}(x)_{n,\lambda} &= \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, (n \geq 0) \\ &= \sum_{k=0}^{n}\sum_{m=0}^{k} S_{2,\lambda}(n,k)S_1(k,m)x^m.\end{aligned} \tag{2}$$

We note that $S_{2,\lambda}(n,k)$ converges to $S_2(n,k)$ and $e_\lambda^x(t)$ converges to e^x if λ tends to 0. There has been various trials for the study degenerate Bell polynomials and numbers. In [9], the first works on the degenerate Bell numbers and polynomials are done by Kims,

$$(1+\lambda)^{\frac{x}{\lambda}(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x)\frac{t^n}{n!}.$$

Later in [11], T.Kim, D. Kim and D. V. Dolgy studied the partially degenerate Bell polynomials

$$e^{x(e_\lambda(t)-1)} = e^{x((1+\lambda t)^{1/\lambda}-1)} = \sum_{n=0}^{\infty} \mathrm{bel}_{n,\lambda}(x)\frac{t^n}{n!}.$$

In [6], D. V. Dolgy et al. defined and studied the fully degenerate Bell polynomials by the generating function

$$e_\lambda(x(e_\lambda(t)-1)) = (1+\lambda x((1+\lambda t)^{1/\lambda}-1))^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \mathrm{B}_{n,\lambda}(x)\frac{t^n}{n!}.$$

From the definition of fully degenerate Bell polynomials and numbers, they gave some interesting identities related with special numbers and polynomials. And recently in [15], S. Pyo and T. Kim studied some identities on the fully degenerate Belly polynomials by using differential equations.

In this paper, we consider new denenerating approach to the Bell polynomials which is called fully degenerate Bell polynomials of the second kind. We study some identities from the generating function and specially by using the differential equations on those polynomials. We derive some identities of our fully degenerate Bell polynomials of the second kind, which related with higher order Daehee polynomials and higher order Bernoulli numbers.

2. Identities for the degenerate Bell polynomials

We define fully degenerate Bell polynomials of the second kind, denoted by $\mathrm{B}^*_{n,\lambda}(x)$ by the generating function

$$(1+\lambda(e_\lambda(t)-1))^{\frac{x}{\lambda}} = (1+\lambda((1+\lambda t)^{1/\lambda}-1))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!}. \tag{3}$$

The generating function of fully degenerate Bell polynomials of the second kind in (3) shows that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathrm{B}^{*}_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0}(1+\lambda(e_\lambda(t)-1))^{\frac{x}{\lambda}}\\ &= \lim_{\lambda\to 0}(1+\lambda((1+\lambda t)^{1/\lambda}-1)^{\frac{x}{\lambda}}\\ &= \sum_{n=0}^{\infty}\mathrm{B}_n(x)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of both sides, we get

$$\lim_{\lambda\to 0}\mathrm{B}^{*}_{n,\lambda}(x) = \mathrm{B}_n(x)$$

and when $x = 1$, $\mathrm{B}^{*}_{n,\lambda} = \mathrm{B}^{*}_{n,\lambda}(1)$ are called the fully degenerate Bell numbers of the second kind. We can see that our degenerate Bell numbers are same that of fully degenerate Bell numbers, which are defined in [6].

From the generating function of the fully degenerate Bell polynomials of the second kind in (3),

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{B}^{*}_{n,\lambda}(x)\frac{t^n}{n!} &= \sum_{l=0}^{\infty}\left(\frac{x}{\lambda}\right)_l\frac{\lambda^l[(1+\lambda t)^{1/\lambda}-1]^l}{l!}\\ &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\sum_{l=0}^{n}S_{2,\lambda}(n,l)\frac{t^n}{n!} \qquad (4)\\ &= \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}(x)_{l,\lambda}S_{2,\lambda}(n,l)\right]\frac{t^n}{n!}.\end{aligned}$$

By (2) and (4), we have the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

(1) $\mathrm{B}^{*}_{n,\lambda}(x) = \sum_{l=0}^{n}(x)_{l,\lambda}S_{2,\lambda}(n,l)$

(2) $\mathrm{B}^{*}_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\sum_{m=0}^{k}S_{2,\lambda}(n,l)S_{2,\lambda}(l,k)S_1(k,m)x^m.$

We note that

$$\frac{1}{l!}[(1+\lambda t)^{1/\lambda}-1]^l = \frac{1}{l!}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(1+\lambda t)^{\frac{m}{\lambda}}.$$

Thus the right hand side of (4), we have

$$\begin{aligned}&\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{1}{l!}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\left(\sum_{n=0}^{\infty}\left(\frac{m}{\lambda}\right)_n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{l}{m}(x)_{l,\lambda}(-1)^{l-m}\frac{1}{l!}(m)_{n,\lambda}.\end{aligned}$$

Thus we have the following

Corollary 2.2. *For $n \geq 0$, we have*

$$(1)\quad \mathrm{B}^*_{n,\lambda}(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(x)_{l,\lambda}\binom{l}{m}(-1)^{l-m}\frac{1}{l!}(m)_{n,\lambda}$$

$$(2)\quad \mathrm{B}^*_{n,\lambda} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(1)_{l,\lambda}\binom{l}{m}(-1)^{l-m}\frac{1}{l!}(m)_{n,\lambda}.$$

For the convenience we denote the generating function for fully degenerate Bell polynomials of the second kind in (4) as follows

$$F(t,x) = \sum_{n=0}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!}. \tag{5}$$

Then the N-th differentiation of the equation (5), we have the following

$$\begin{aligned}\frac{\partial^N}{\partial t^N}F(t,x) &= \frac{\partial^N}{\partial t^N}\left(\sum_{n=0}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!}\right)\\ &= \sum_{n=N}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^{n-N}}{(n-N)!}\\ &= \sum_{n=0}^{\infty}\mathrm{B}^*_{n+N,\lambda}(x)\frac{t^n}{n!}.\end{aligned} \tag{6}$$

The following (7) and (8) are adopted from [15]

$$\begin{aligned}\frac{1}{1+\lambda(e_\lambda(t)-1)} &= \sum_{j=0}^{\infty}(-1)^j\lambda^j j!\frac{(e_\lambda(t)-1)^j}{j!}\\ &= \sum_{j=0}^{\infty}(-1)^j\lambda^j j!\sum_{m=j}^{\infty}S_{2,\lambda}(m,j)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^j\lambda^j j!S_{2,\lambda}(m,j)\frac{t^m}{m!}.\end{aligned} \tag{7}$$

And

$$(1+\lambda t)^{\frac{1}{\lambda}-1} = \sum_{k=0}^{\infty}(1)_{k+1,\lambda}\frac{t^k}{k!}. \tag{8}$$

By using (7) and (8), we have

$$\begin{aligned}\frac{\partial F}{\partial t} &= \left(x\sum_{l=0}^{\infty}\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^j\lambda^j j!S_{2,\lambda}(m,j)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}(1)_{k+1,\lambda}\frac{t^k}{k!}\right)\\ &= \left(x\sum_{l=0}^{\infty}\mathrm{B}^*_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{m}{k}(-1)^j\lambda^j j!S_{2,\lambda}(k,j)(1)_{m-k-1,\lambda}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{k}x\,\mathrm{B}^*_{n-m,\lambda}(x)(-1)^j\lambda^j j!(1)_{m-k-1,\lambda}S_{2,\lambda}(k,j)\right]\frac{t^n}{n!}.\end{aligned} \tag{9}$$

From the equations (6) and (9), we have the following theorem.

Theorem 2.3. *For any $\lambda \in \mathbb{R}$ and non-negative integer n, we have*

$$\mathrm{B}^*_{n+1,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{k}\mathrm{B}^*_{n-m,\lambda}(x)(-1)^j\lambda^j j! x(1)_{m-k-1,\lambda}S_{2,\lambda}(k,j).$$

Let λ approaches to zero in the Theorem 2.3, then only $j=0$ is possible from $\lambda^j = 0$ and we get $k=0$, from $S_{2,\lambda}(k,j) = S_{2,\lambda}(k,0) = 1$. Otherwise $S_{2,\lambda}(k,0) = 0$ if $k > 1$. Thus we get the following well-known identity in [6, 15]

Corollary 2.4. *For any $\lambda \in \mathbb{R}$ and non-negative integer n, we have*

$$\mathrm{B}_{n+1}(x) = x\sum_{m=0}^{n}\binom{n}{m}\mathrm{B}_m(x).$$

Furthermore, let replace t by $\log_\lambda(1+t) = \frac{(1+t)^\lambda - 1}{\lambda}$ in (3), we have

$$\sum_{m=0}^{\infty}\mathrm{B}^*_{m,\lambda}(x)\frac{(\log_\lambda(1+t))^m}{m!} = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}. \tag{10}$$

The left hand side of (10),

$$\sum_{m=0}^{\infty}\mathrm{B}^*_{m,\lambda}(x)S_{1,\lambda}(n,m)\frac{t^m}{m!} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}.$$

Theorem 2.5. *For any $\lambda \in \mathbb{R}$ and non-negative integer n, we have*

$$(x)_{n,\lambda} = S_{1,\lambda}(n,m)\,\mathrm{B}^*_{m,\lambda}(x).$$

When $\lambda \to 0$ in the above Theorem 2.5, we have the well known identity in [6, 15]

$$x^n = S_1(n,m)\,\mathrm{B}_m(x).$$

For the inversion formula of (10), from the identity,

$$\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} = (1+\lambda t)^{\frac{x}{\lambda}}.$$

Let t be replaced by $(1+\lambda t)^{\frac{x}{\lambda}} - 1$, then

$$\begin{aligned}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{\left((1+\lambda t)^{\frac{x}{\lambda}}-1\right)^n}{m!} &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}S_{1,\lambda}(n,m)(x)_{m,\lambda}\frac{t^n}{n!}\\ &= \left(1+\lambda((1+\lambda t)^{\frac{1}{\lambda}}-1)\right)^{\frac{x}{\lambda}}\\ &= \sum_{n=0}^{\infty}\mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus we have the inversion formula for the Theorem 2.5

Theorem 2.6. *For any $\lambda \in \mathbb{R}$ and non-negative integer n, we have*

$$\mathrm{B}^*_{n,\lambda}(x) = \sum_{m=0}^{n}S_{2,\lambda}(n,m)(x)_{m,\lambda}.$$

When $\lambda \to 0$ in the above Theorem 2.6, we have the well known identity

$$\mathrm{B}_n(x) = \sum_{m=0}^{n} S_2(n,m)(x)_m.$$

3. Differential equations using the fully degenerate Bell polynomials of the second kind

For the convenience of this paper, we denote

$$\begin{aligned} T &= (1+\lambda t)^{1/\lambda} - 1 = e_\lambda(t) - 1 \\ \text{and} \quad F &= F(t,x) = (1+\lambda T)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!}. \end{aligned} \tag{11}$$

From (2) and (11), we see the N-th differentiation of the F with respect to x

$$\begin{aligned} \frac{\partial^N F}{\partial x^N} &= \frac{\partial^N}{\partial x^N}\sum_{n=0}^{\infty} \mathrm{B}^*_{n,\lambda}(x)\frac{t^n}{n!} \\ &= \sum_{n=N}^{\infty}\left(\frac{\partial^N}{\partial x^N}\mathrm{B}^*_{n,\lambda}(x)\right)\frac{t^n}{n!}. \end{aligned} \tag{12}$$

We observe that

$$\frac{\partial F}{\partial x} = (1+\lambda T)^{\frac{x}{\lambda}}\frac{\log(1+\lambda T)}{\lambda} = F\frac{\log(1+\lambda T)}{\lambda}.$$

Inductively, we get for $N \geq 1$,

$$\frac{\partial^N F}{\partial x^N} = F^{(N)} = F\left(\frac{\log(1+\lambda T)}{\lambda}\right)^N. \tag{13}$$

The Daehee polynomials are defined by the generating function

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}.$$

And the Daehee polynomials of order r are defined by the generating function in [16, 17],

$$\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}.$$

We express $F^{(N)}$ via Daehee polynomials of order N, let's consider the following

$$
\begin{aligned}
F^{(N)} &= F\left(\frac{\log(1+\lambda T)}{\lambda}\right)^{N} \\
&= \left(\frac{\log(1+\lambda T)}{\lambda T}\right)^{N}(1+\lambda T)^{\frac{x}{\lambda}}T^{N} \\
&= \left[\sum_{m=0}^{\infty} D_{m}^{(N)}\left(\frac{x}{\lambda}\right)\frac{\lambda^{m}T^{m}}{m!}\right]T^{N} \\
&= \sum_{m=0}^{\infty} D_{m}^{(N)}\left(\frac{x}{\lambda}\right)\frac{\lambda^{m}}{m!}T^{m+N} \\
&= \sum_{m=0}^{\infty} D_{m}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^{m}(m+N)_{N}\sum_{n=m+N}^{\infty} S_{2,\lambda}(n,m+N)\frac{t^{n}}{n!} \\
&= \sum_{n=N}^{\infty}\left[\sum_{m=0}^{l-N} D_{m}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^{m}(m+N)_{N}S_{2,\lambda}(n,m+N)\right]\frac{t^{n}}{n!}.
\end{aligned}
\tag{14}
$$

Thus by (12) and (14), we have

Theorem 3.1. *For any $\lambda \in \mathbb{R}$ and non-negative integers n, N with $n \geq N$, we have*

$$
\frac{\partial^{N}}{\partial x^{N}}\,\mathrm{B}^{*}_{n,\lambda}(x) = \sum_{m=0}^{l-N} D_{m}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^{m}(m+N)_{N}S_{2,\lambda}(n,m+N).
$$

The following degenerate Bernoulli numbers of order r, $\beta_{n,\lambda}^{(r)}$ are very well known in [3, 12]

$$
\left(\frac{t}{e_{\lambda}(t)-1}\right)^{r} = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}\frac{t^{n}}{n!}. \tag{15}
$$

Now we relate $\frac{\partial^{N}}{\partial x^{N}}\mathrm{B}^{*}_{n,\lambda}(x)$ with higher order degenerate Bernoulli numbers and higher order Daehee polynomials. We consier $F^{(N)}$ in (13) as follows

$$
\begin{aligned}
F^{(N)} &= F\left(\frac{\log(1+\lambda T)}{\lambda}\right)^{N} \\
&= \left(\frac{\log(1+\lambda T)}{\lambda T}\right)^{N}(1+\lambda T)^{\frac{x}{\lambda}}T^{N}.
\end{aligned}
$$

Then we can rewrite

$$
F^{(N)}t^{N}T^{-N} = \left(\frac{\log(1+\lambda T)}{\lambda T}\right)^{N}(1+\lambda T)^{\frac{x}{\lambda}}t^{N}. \tag{16}
$$

Thus we have

$$\begin{aligned} F^{(N)}t^N T^{-N} &= t^N \sum_{m=0}^{\infty} D_{m,\lambda}^{(N)}\left(\frac{x}{\lambda}\right)\frac{\lambda^m T^m}{m!} \\ &= \sum_{m=0}^{\infty} D_{m,\lambda}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^m \sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} D_{m,\lambda}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^m S_{2,\lambda}(n,m)\frac{t^{n+N}}{n!} \\ &= \sum_{n=N}^{\infty}\sum_{m=0}^{n-N} D_{m,\lambda}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^m S_{2,\lambda}(n-N,m)(n+N)_N\frac{t^n}{n!}. \end{aligned} \tag{17}$$

On the other hand, by (15), we have

$$t^N T^{-N} = \left(\frac{t}{e_\lambda(t)-1}\right)^N = \sum_{m=0}^{\infty}\beta_{m,\lambda}^{(N)}\frac{t^m}{m!}$$

and the left hand side of (16)

$$\begin{aligned} F^{(N)}t^N T^{-N} &= \sum_{n=N}^{\infty}\left(\frac{\partial^N}{\partial x^N}\,\mathrm{B}_{n,\lambda}^*(x)\right)\frac{t^n}{n!}\sum_{m=0}^{\infty}\beta_{m,\lambda}^{(N)}\frac{t^m}{m!} \\ &= \sum_{n=N}^{\infty}\sum_{m=0}^{n-N}\binom{n}{m}\frac{\partial^N}{\partial x^N}\,\mathrm{B}_{n,\lambda}^*(x)\beta_{n-m,\lambda}^{(N)}\frac{t^n}{n!}. \end{aligned} \tag{18}$$

By (17) and (18), we have the following identity.

$$\sum_{m=0}^{n-N}\binom{n}{m}\frac{\partial^N}{\partial x^N}\,\mathrm{B}_{n,\lambda}^*(x)\beta_{n-m,\lambda}^{(N)} = \sum_{m=0}^{n-N} D_{m,\lambda}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^m S_{2,\lambda}(n-N,m)(n+N)_N.$$

Theorem 3.2. *For any $\lambda \in \mathbb{R}$ and non-negative integers n, N with $n \geq N$, we have*

$$\sum_{m=0}^{n-N}\binom{n}{m}\frac{\partial^N}{\partial x^N}\,\mathrm{B}_{n,\lambda}^*(x)\beta_{n-m,\lambda}^{(N)} = \sum_{m=0}^{n-N} D_{m,\lambda}^{(N)}\left(\frac{x}{\lambda}\right)\lambda^m S_{2,\lambda}(n-N,m)(n+N)_N.$$

On the other hand,

$$\begin{aligned} F^{(N)} &= F\left(\frac{\log(1+\lambda T)}{\lambda}\right)^N \\ &= F\frac{N!}{\lambda^N}\frac{(\log(1+\lambda T))^N}{N!}. \end{aligned}$$

$$
\begin{aligned}
\sum_{n=N}^{\infty} \frac{\partial^N \mathrm{B}^*_{n,\lambda}(x)}{\partial x^N}\frac{t^n}{n!} &= F\frac{N!}{\lambda^N}\sum_{l=N}^{\infty} S_1(l,N)\frac{\lambda^l T^l}{l!} \\
&= (1+\lambda T)^{\frac{x}{\lambda}}\frac{N!}{\lambda^N}\sum_{l=N}^{\infty} S_1(l,N)\frac{\lambda^l T^l}{l!} \\
&= \sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{T^m}{m!}\frac{N!}{\lambda^N}\sum_{l=N}^{\infty} S_1(l,N)\frac{\lambda^l T^l}{l!} \\
&= \sum_{k=N}^{\infty}\sum_{l=0}^{k-N}\binom{k}{l}(x)_{k-l,\lambda}\frac{\lambda^l N!}{\lambda^N}S_1(l,N)\frac{T^k}{k!} \\
&= \sum_{k=N}^{\infty}\sum_{l=0}^{k-N}\binom{k}{l}(x)_{k-l,\lambda}\frac{\lambda^n N!}{\lambda^N}S_1(l,N)\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} \\
&= \sum_{n=N}^{\infty}\left[\sum_{k=N}^{n-N}\sum_{l=0}^{k-N}\binom{k}{l}(x)_{k-l,\lambda}\frac{\lambda^n N!}{\lambda^N}S_1(n,m)S_{2,\lambda}(n,k)\right]\frac{t^n}{n!}.
\end{aligned} \tag{19}
$$

Therefore by the comparing the coefficients of (19), we have the following identity.

Theorem 3.3. *For any $\lambda \in \mathbb{R}$ and non-negative integers n, N with $n \geq N$, we have*

$$
\frac{\partial^N}{\partial x^N}\mathrm{B}^*_{n,\lambda}(x) = \sum_{k=N}^{n-N}\sum_{l=0}^{k-N}\binom{k}{l}(x)_{k-l,\lambda}\frac{\lambda^n N!}{\lambda^N}S_1(n,m)S_{2,\lambda}(n,k).
$$

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